K3 Compactifications and a Moonshine for M_{24}

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Outline

- Motivations
- Review: K3 Compactifications
- M_{24} Moonshines
    - Interlude: Monstrous Moonshine
    - 1/4-BPS Moonshine
    - 1/2-BPS Moonshine

I. Motivations

for studying K3 compactification of string theory

1. Gravity Beyond GR

Uniquely precise and exact testing ground for new ideas.
eg1. higher-order corrections
[Cardoso, Kappeli, Mohaupt, de Wit '04, David, Jatkar, Sen 05,06, Kraus-Larsen 05]
eg2. Prescription Euclidean path integral (Quantum Entropy Functions)
[A. Sen 09]

2. Microscopic Theory of BPS States

Microscopic theory of black holes
= Computing spectrum of non-pert. BPS states
= Quantising D-brane moduli space
= counting of sLags

3. Wall-Crossing Physics

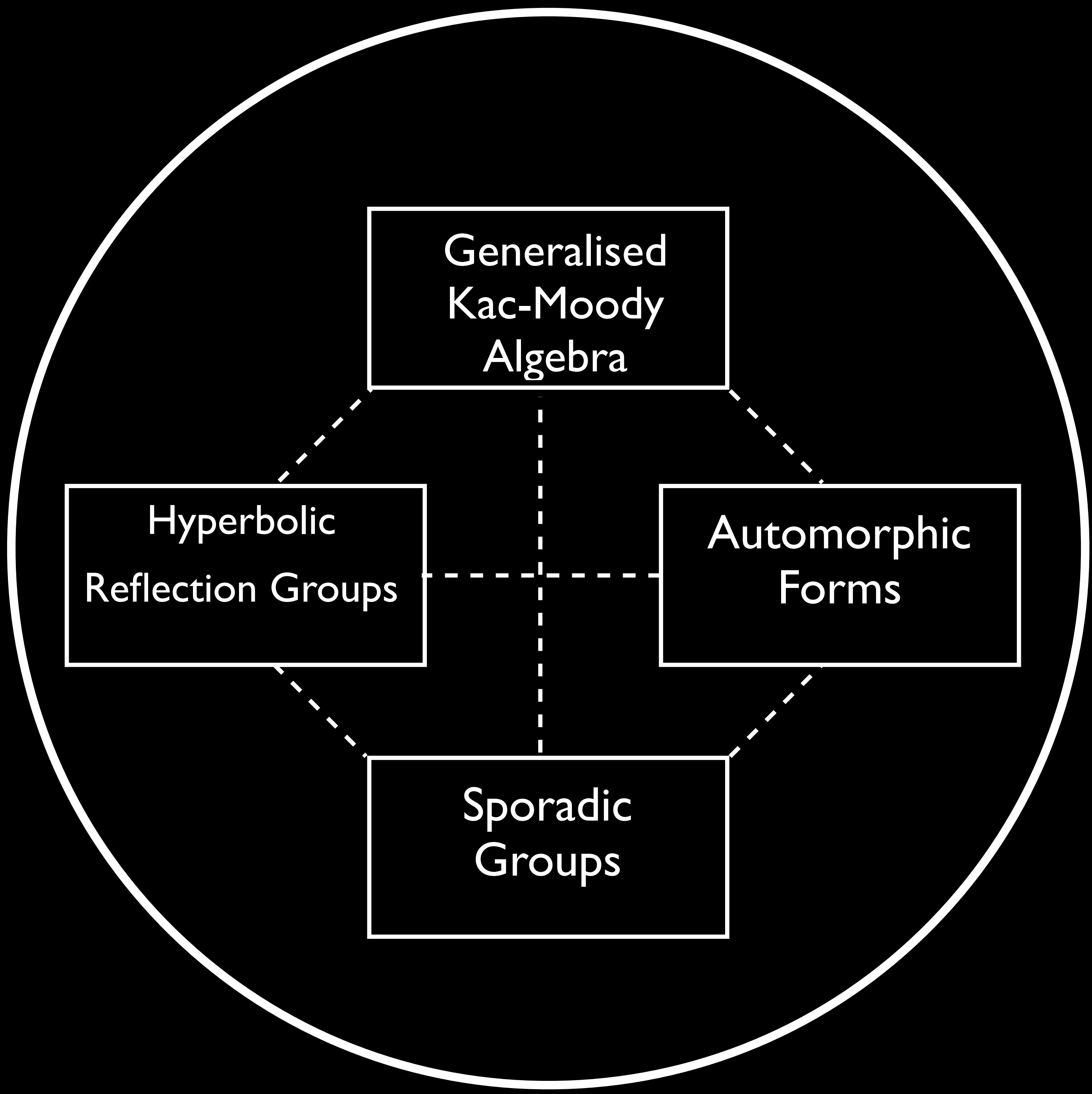
Change the parameters in the Lagrangian of the theory. In general the spectrum changes => chamber structure

4. Symmetries of K3 Compactifications

K3 ubiquitous in string theory: dualities, model building, geometric engineering....

5. Mathematical Motivations

Discrete Mathematics



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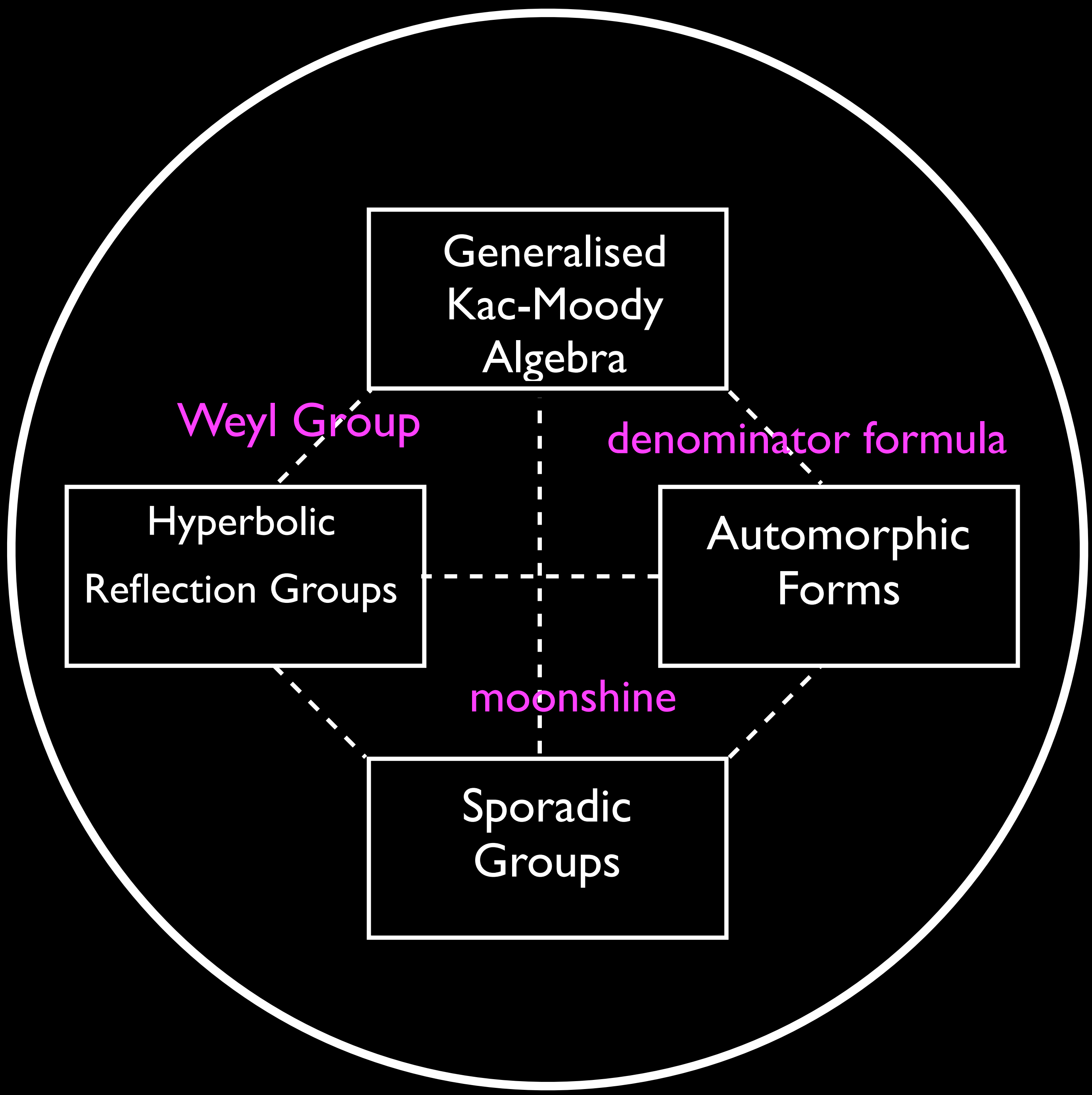

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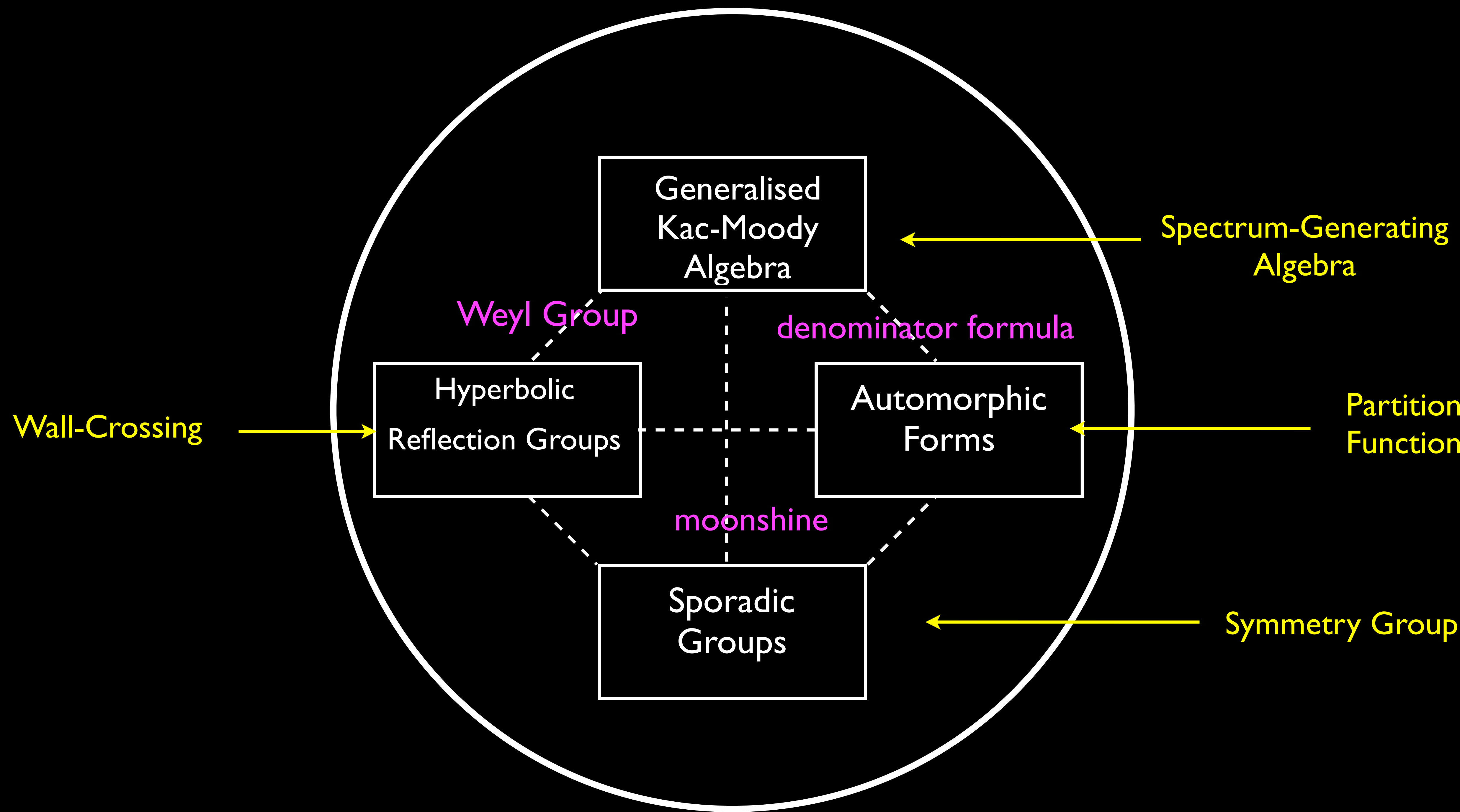

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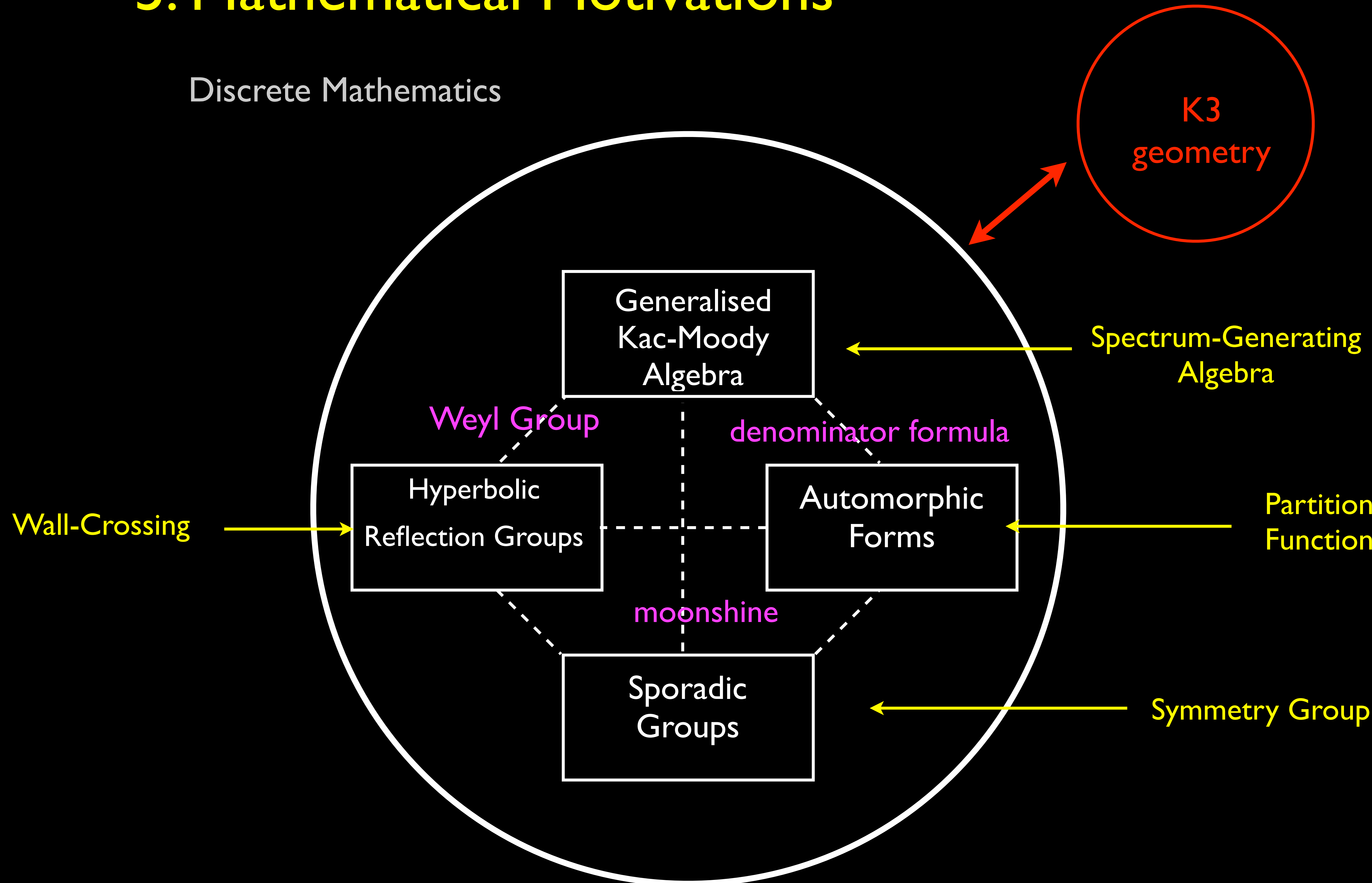

II. Review: K3 Compactifications

Partial List of Contributors....

N. Banerjee, Cardoso,----, Dabholkar, David, Dijkgraaf, Eguchi, Gaiotto, Gomes, Govindarajan, Jatkar, Kappeli, Kawai, Krishna, Mukherjee, Mukhi, Murhty, Nampuri, Nigam, Pioline, Sen, Strominger, Srivastava, Verlinde*2, de Wit, Yamada, Yang, Yin, ...

N=(2,2) SCFT and Elliptic Genus of Calabi-Yau manifolds

N=(2,2) 2d sigma model on

$$\mathcal{L} \sim \int_{\Sigma} \left(|\partial\phi|^2 - B\right) + \text{fermions}$$

when the target space is CY,
conformal with conserved currents

$$J, G^{\pm}, T$$

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Define the *Elliptic Genus* of X

$$Z(\tau, z; X) = \mathrm{Tr}_{RR}\Big((-1)^{J_L + J_R} y^{2\pi i J_L} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\Big)$$

$$q = e^{2\pi i \tau}, y = e^{2\pi i z}$$

Elliptic Genus of CY:

- Generalisation of various topological invariants e.g.

$$Z(\tau, z; X)\Big|_{z=0} = \chi(X)$$
$$Z(\tau, z; X)\Big|_{z=1/2} = \sigma(X)$$

- Transforms nicely (weak Jacobi form) under *SL(2,Z)*

- These two conditions are often enough to determine $Z(\tau, z; X)$ uniquely.

Elliptic Genus of K3 surfaces:

$$
\begin{aligned}
Z(\tau, z; K3) &= (2y + 20 + 2y^{-1}) + q(\dots) + q^2(\dots) \\
&= \sum_{n \geq 0, \ell \in \mathbb{Z}} c(4n - \ell^2) q^n y^\ell
\end{aligned}
$$

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Moreover,

K3 is also hyper-Kähler $\longrightarrow$

N=(4,4) superconformal symmetry $\longrightarrow$

The elliptic genus can be decomposed into characters of representations of N=4 superconformal algebra [Eguchi-Ooguri-Taormina-Yang '89]

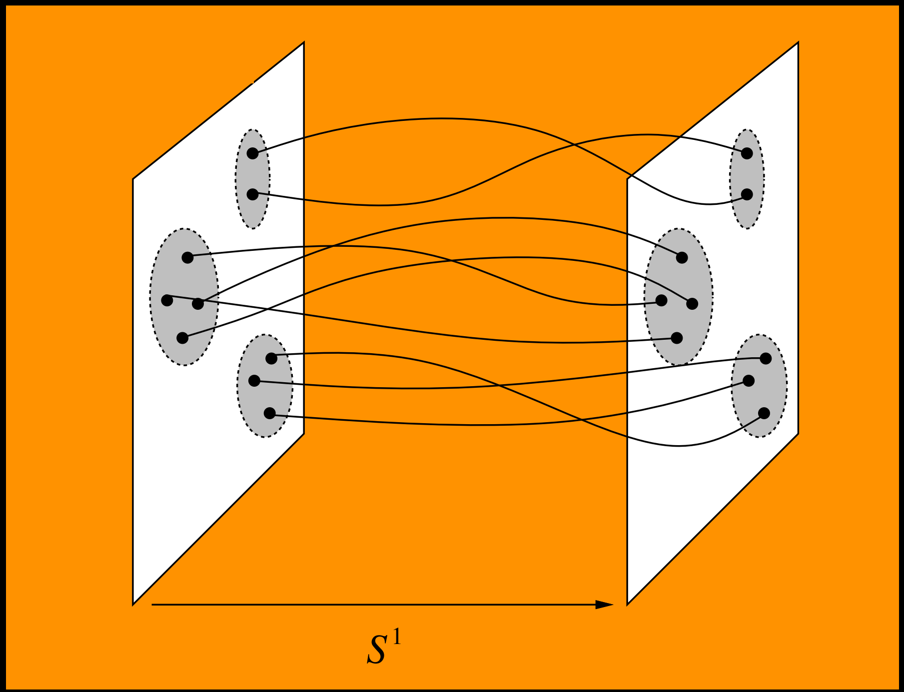
Elliptic Genus of Symmetric Product $S^N X = X^N / S_N$:

$$\sum_N p^N Z(\tau, z; S^N X) \quad = \quad \text{2nd quantized string partition function}$$

$$\text{on } X \times S^1$$

$$= \quad \prod_{n,m,\ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(4nm-\ell^2)}}$$

Counting susy ground states of D1-D5 string on *K3XS[1]* with *N=Q1Q5+1*



[Dijkgraaf-Moore-Verlinde[2] '97]

Microscopic Spectrum: the 1/4-BPS States in type II on K3xT^2

States preserving 1/4 of supersymmetries can be realized as D1-D5-P-Taub-NUT bound states in type IIB frame, with partition function

$$\frac{1}{\Phi(\Omega)} = \frac{1}{pqy} \prod_{n,m,\ell} \frac{1}{(1-p^n q^m y^\ell)^{c(4nm-\ell^2)}}$$

$$\Phi(\Omega)^{1/2} = e(-\varrho) \prod_{\alpha\in\Delta_+} \big(1-e(-\alpha)\big)^{\mathrm{mult}\,\alpha}$$

= denominator formula of a
generalised Kac-Moody algebra (the dyon algebra)

Microscopic Spectrum: the 1/2-BPS States in type II on K3xT^2

States preserving 1/2 of supersymmetries can be realized as perturbative heterotic string states, with partition function

$$\frac{1}{\eta^{24}(\tau)} = \frac{1}{q\prod_{n\geq 1}(1-q^n)^{24}}$$

= a weight 12 modular (cusp) form.

III. M_{24} Moonshines

Based on
M.C. , Eguchi-Ooguri-Tachikawa, Eguchi-Hikami, Gaberdiel-Hohenegger-Volpano, Govindarajan 10

K3 Elliptic Genus

$$Z(\tau, z; K3) = 8 \sum_{i=2}^{4} \Big(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)}\Big)^2$$

$$= \sum_{\substack{n \geq 0 \\ n, \ell \in \mathbb{Z}}} c(4n - \ell^2) q^n y^\ell$$

$$= 20\, ch_{1/4,0}(\tau, z) - 2\, ch_{1/4,1/2}(\tau, z) + \sum_{n \geq 1} t_n\, ch_{n+1/4,1/2}(\tau, z)$$

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in terms of N=4 SCA characters

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\end{aligned}
$$

in terms of N=4 SCA characters

$$
T(\tau) = \sum_{n=1}^{\infty} t_n q^n = 2\{ 45 q + 231 q^2 + 770 q^3 + 2277 q^4 + 5796 q^5 + \dots \}
$$

dim. of irrep of M24

(EOT '10)

Q: What does it mean?
What do we do with this?

III-i. Interlude:
Monstrous Moonshine

$|\mathbb{M}| \sim 8 \times 10^{53}$

194 conjugacy classes ($[hgh^{-1}] = [g]$)

$\Leftrightarrow$ 194 irreps

Klein inv $\quad J(\tau) = j(\tau) - 744 = \sum c(n) q^n$

$$= q^{-1} + \underset{\substack{\| \\ 1+196883}}{196884}\, q + \underset{\substack{\| \\ 1+196883+21296876}}{21493760}\, q^2 + \cdots$$

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as if there exists an ∞-dim $\mathbb{Z}$-graded $\mathbb{M}$-representation

$$V^\natural = V^\natural_{-1} \oplus V^\natural_1 \oplus V^\natural_2 \oplus \ldots$$

such that $\dim V^\natural_n = c(n)$

if true, can also consider "McKay-Thompson" series

$$J_g(\tau) = \sum_n q^n \mathrm{Tr}_{V_n^\natural}(g) = \sum_n c_g(n) q^n \quad \forall\, [g] \in \mathbb{M}$$

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Moonshine Conjecture (Conway-Norton '79):

All $J_g(\tau)$are invariant under some modular group $\Gamma_g \subset SL(2,\mathbb{R})$

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Q: Why are sporadic groups related to modular forms?

Answer: CFT!

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’88 Frenkel-Lepowsky-Meurmann (see also Dixon-Ginsparg-Harvey)

$c=24$ chiral CFT

bosonic str. on $\Lambda_{\text{Leech}}/\mathbb{Z}_2$

$\circlearrowleft$

$\mathbb{M}$

$$V^\natural = \text{Hilbert spc} = V^\natural_{-1} \oplus V^\natural_0 \oplus V^\natural_1 \oplus \cdots$$

$$L_0 - c/24$$

$$\mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24} = \sum{}' q^n \dim V^\natural_n \stackrel{?}{=} J(\tau)$$

(Partial) Answer: CFT!

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$V^\natural$ = Hilbert spc $= V^\natural_{-1} \oplus V^\natural_0 \oplus V^\natural_1 \oplus \cdots$

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'88 Borcherds' proof and the invention of generalised Kac-Moody algebras

$V^\natural \xrightarrow{①} \mathfrak{m} \xrightarrow{②}$ (twisted) denominator id $\longrightarrow$ compare 12 coeff. for each J_g

(I) Monster algebra $\mathfrak{m}$

GKM has the usual triangular decomposition

$\{f_\alpha, e_\alpha, h_\alpha\} \quad , \alpha \in \Delta = \text{root system}$

root lattice $= \Gamma^{1,1} \quad , \alpha = (m, n)\,,\ |\alpha|^2 = mn$

root space $\mathcal{V}_\alpha \simeq V^\natural_{|\alpha|^2}$

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(2) Twisted Denominator Formula

$\dim V^\natural_{|\alpha|^2} =$ root multiplicity

$$j(p) - j(\sigma) = \left(\frac{1}{p} - \frac{1}{q}\right) \prod_{n,m>0} (1 - p^n q^m)^{c_{nm}}$$

"

$$\begin{cases} p = e^{2\pi i \rho} \\ q = e^{2\pi i \sigma} \end{cases}$$

twist:

$$\sum_{m\in\mathbb{Z}} (\mathrm{Tr}_{V^\natural_m}\, g)\, p^m - \sum_{n\in\mathbb{Z}} (\mathrm{Tr}_{V^\natural_n}\, g)\, q^n$$

$$= p^{-1} \exp\left\{ - \sum_{k>0} \sum_{\substack{m>0 \\ n\in\mathbb{Z}}} \frac{(p^m q^n)^k}{k}\, \mathrm{Tr}_{V^\natural_{nm}}\, g^k \right\}$$

III-ii. 1/4-BPS (elliptic genus) Moonshine

K3 Elliptic Genus

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dim. of irrep of M_{24}

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Question:

Does there exists an ∞-dim $\mathbb{Z}$-graded M_{24}-representation

$K^\natural = K_1^\natural \oplus K_2^\natural \oplus \ldots$, such that the following is true?

$$Z(\tau, z; K3) = \frac{\vartheta_1^2}{\eta^3}\left\{ \dim(\rho_1 \oplus \rho_{23})\, \mu(\tau, z) + q^{-1/8}\left(-2 + \sum q^n \dim K_n^\natural\right)\right\}$$

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Or: Is $\mathcal{H}$ a (∞-dim) representation ofM_{24}?

Two Consequences if true (another one will come later):

① $Z_g(\tau, z; K3) = \mathrm{Tr}_R(g \cdot \cdots\cdots) = {}_{g}\square_{1}$

$\forall\, g \in M_{24}$

$$= \sum c_g(4n-\ell^2)\, q^n y^\ell$$

$$= \frac{\vartheta^2}{\eta^3}\left\{ \left(\mathrm{Tr}_{\rho_1 \oplus \rho_2}\, g\right) \mu(\tau, z) + q^{-1/8}\left(-2 + \sum q^n\, \mathrm{Tr}_{K_n}\, g\right)\right\}$$

② moreover, from

$$\begin{cases} {}_{g}\square_{h} \longmapsto {}_{g^a h^b}\square_{g^c h^d} \\ \tau \longmapsto \dfrac{a\tau+b}{c\tau+d} \end{cases}$$

$$Z_g\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}; K3\right) = e^{i\theta_g(\gamma)}\, Z_g(\tau, z; K3)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathrm{ord}(g))$

So Far So Good (has to be true!):

Candidates for $Z_g(\tau, z; K3)$ found for all $g \in M_{24}$.

$K_n^\natural$ found for $n \leq 600$.

[M.C. , Eguchi-Ooguri-Tachikawa, Eguchi-Hikami, Gaberdiel-Hohenegger-Volpano 2010]

III-iii. 1/2-BPS Moonshine

An older M_{24} Moonshine (G. Mason '85)

An older M24 Moonshine (G. Mason '85)

$[g] \mapsto \eta_g(\tau)$ for all $g \in M_{24}$

$$
\begin{aligned}
& e.g. \\
& \eta_{1A}(\tau) = \eta^{24}(\tau) \\
& \eta_{2A}(\tau) = \eta^{8}(\tau)\eta^{8}(2\tau) \\
& \quad \vdots
\end{aligned}
$$

Two Versions of M_{24} Moonshine?!
Unrelated to K3?

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NO!

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NO!

1/2-BPS partition function of K3xT^2 compactified type II string

$$\frac{1}{\eta^{24}(\tau)} = \frac{1}{q\prod_{n\geq 1}(1-q^n)^{24}}$$

more generally, twisted 1/2-BPS partition function

$$\eta_g(\tau)?$$

Consistent with all examples we know!

[Govindarajan-Krishna 09, A. Sen 05]

Even Better: only ONE moonshine after all!

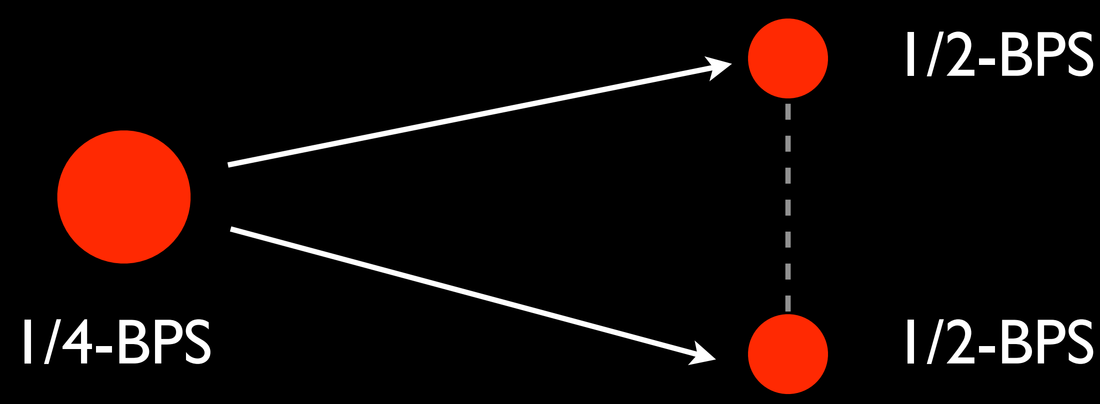
The elliptic genus moonshine implies:
the harmonic oscillators generating 1/4-BPS spectrum
(or roots of dyon algebra) furnish M_{24} reps'.

Under this assumption I can compute the
"twisted" 1/4-BPS partition function
(twisted denominator formula)

$$1/\Phi_g(\Omega)$$

[M.C.'10]

But 1/4-BPS spectrum must know about the 1/2-BPS one because of Wall-Crossing!



Manifested in the poles of the partition function. Now generalise this idea to twisted ones. Indeed we find:

$$\lim_{\nu\to 0} \frac{1}{\Phi_g(\Omega)} \sim \frac{1}{\nu^2}\frac{1}{\eta_g(\rho)}\frac{1}{\eta_g(\sigma)} \qquad \forall g \in M_{24}$$

[M.C.'10]

IV. Conclusions and Open Questions

STATUS

- Exciting new symmetries!
- It all looks right so far but nothing is proven. Not even a “physicists’ proof”.

To Do

- Understand special symmetries of K3 better: mirror symmetries, B-field.
- Understanding the dyon algebra better. This should be the key to a final proof.
- Study the new orbifold theory.
- ......
- Many many more for mathematicians.

CFTs, black holes, wall-crossing, geometry, algebra, number theory all in one....

WELCOME TO THE MOON AND JOIN THE K3 PARADISE!

Thank you!

M_{24} and Classical Geometry of K3 (Mukai, Kondo)

$M_{24} \subset S_{24}$

sym. of Golay code $\mathcal{G}$, code word $\in \mathbb{Z}_2^{24}$

$\to$ Niemeier lattice (even, self-dual 24 dim)

$A_1^{24} \subset N \subset A_1^{*24}$

$A_1^{*24}/A_1^{24} \simeq \mathbb{Z}_2^{24}$, $N/A_1^{24} \simeq \mathcal{G}$

$L = H^*(K3, \mathbb{Z}) \simeq \Gamma^{4,20}$.

G: symplectic automorphism of K3 ((0,2)-form inv.)

$L^G \subset L$: fixed pnt lattice, containing at least $H^{2,0}, H^{0,2}$, $H^0, H^4, J \in H^{1,1}$

$L_G \subset L$: orthogonal lattice

(Nikulin) $L_G \overset{\text{primitive}}{\hookrightarrow} N$

$\Rightarrow G \subset M_{24}$

but G is necessarily smaller than M_{24} since it has $\dim L^G \geq 5$